8-1 Multiplying and Dividing Rational Expressions

Main Ideas

- Simplify rational expressions.
- Simplify complex fractions.

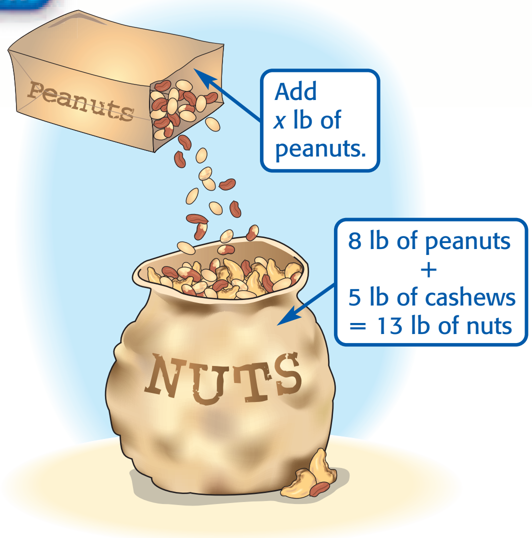
New Vocabulary

rational expression

complex fraction

GET READY for the Lesson

The Goodie Shoppe sells candy and nuts by the pound. One item is a mixture made with 8 pounds of peanuts and 5 pounds of cashews. Therefore, $\frac{8}{8+5}$ or $\frac{8}{13}$ of the mixture is peanuts. If the store manager adds an additional x pounds of peanuts to the mixture, then $\frac{8+x}{13+x}$ of the mixture will be peanuts.



Simplify Rational Expressions A ratio of two polynomial expressions such as $\frac{8+x}{13+x}$ is called a **rational expression**. Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar.

To write a fraction in simplest form, you divide both the numerator and denominator by their greatest common factor (GCF). To simplify a rational expression, you use similar techniques.

EXAMPLE 1 Simplify a Rational Expression

a. Simplify $\frac{2x(x-5)}{(x-5)(x^2-1)}$.

Look for common factors.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x}{x^2-1} \cdot \frac{\overset{1}{\cancel{x-5}}}{\underset{1}{\cancel{x-5}}}$$ How is this similar to simplifying $\frac{10}{15}$?

$$= \frac{2x}{x^2-1}$$ Simplify.

b. Under what conditions is this expression undefined?

Just as with a fraction, a rational expression is undefined if the denominator is equal to 0. To find when this expression is undefined, completely factor the original denominator.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x(x-5)}{(x-5)(x-1)(x+1)}$$ $x^2 - 1 = (x-1)(x+1)$

The values that would make the denominator equal to 0 are 5, 1, or −1. So the expression is undefined when $x = 5$, $x = 1$, or $x = -1$.

Study Tip

Excluded Values

Numbers that would cause the expression to be undefined are called **excluded values**.

CHECK Your Progress

Simplify each expression. Under what conditions is the expression undefined?

1A. $\dfrac{3y(y+6)}{(y+6)(y^2-8y+12)}$ **1B.** $\dfrac{4x^3(x^2-7x-8)}{12x(x^2-64)}$

STANDARDIZED TEST PRACTICE — Use the Process of Elimination

2 **For what value(s) of x is $\dfrac{x^2+x-12}{x^2+7x+12}$ undefined?**

A $-4, -3$ **B** -4 **C** 0 **D** $-4, 3$

Test-Taking Tip

Eliminating Choices Sometimes you can save time by looking at the possible answers and eliminating choices, rather than actually evaluating an expression or solving an equation.

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

Notice that if x equals 0 or a positive number, $x^2 + 7x + 12$ must be greater than 0. Therefore, you can eliminate choices C and D. Since both A and B contain -4, determine whether the denominator equals 0 when $x = -3$.

$x^2 + 7x + 12 = (-3)^2 + 7(-3) + 12$ $x = -3$

$= 9 - 21 + 12$ or 0 Multiply and simplify.

Since the denominator equals 0 when $x = -3$, the answer is A.

CHECK Your Progress

2. For what values of x is $\dfrac{x^2+9}{x^2+15x-34}$ undefined?

F $-17, -2$ **G** $-17, 2$ **H** $-2, 17$ **J** $2, 17$

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Sometimes you can factor out -1 in the numerator or denominator to help simplify rational expressions.

EXAMPLE Simplify by Factoring Out –1

3 **Simplify** $\dfrac{z^2w-z^2}{z^3-z^3w}$.

$\dfrac{z^2w-z^2}{z^3-z^3w} = \dfrac{z^2(w-1)}{z^3(1-w)}$ Factor the numerator and the denominator.

$= \dfrac{\overset{1}{\cancel{z^2}}(-1)\overset{1}{\cancel{(1-w)}}}{\underset{z}{\cancel{z^3}}\underset{1}{\cancel{(1-w)}}}$ $w - 1 = -(-w + 1)$ or $-1(1 - w)$

$= \dfrac{-1}{z}$ or $-\dfrac{1}{z}$ Simplify.

CHECK Your Progress

Simplify each expression.

3A. $\dfrac{xy-3x}{3x^2-x^2y}$ **3B.** $\dfrac{2x-x^2}{x^2y-4y}$

Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or reciprocal, of the divisor.

Multiplication

$$\frac{5}{6} \cdot \frac{4}{15} = \frac{\overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{2}} \cdot 2}{\underset{1}{\cancel{2}} \cdot 3 \cdot 3 \cdot \underset{1}{\cancel{5}}} = \frac{2}{3 \cdot 3} \text{ or } \frac{2}{9}$$

Division

$$\frac{3}{7} \div \frac{9}{14} = \frac{3}{7} \cdot \frac{14}{9} = \frac{\overset{1}{\cancel{3}} \cdot 2 \cdot \overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}} \cdot \underset{1}{\cancel{3}} \cdot 3} \text{ or } \frac{2}{3}$$

The same procedures are used for multiplying and dividing rational expressions.

KEY CONCEPT — *Rational Expressions*

Multiplying Rational Expressions

Words To multiply two rational expressions, multiply the numerators and the denominators.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, if $b \neq 0$ and $d \neq 0$.

Dividing Rational Expressions

Words To divide two rational expressions, multiply by the reciprocal of the divisor.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, if $b \neq 0$, $c \neq 0$ and $d \neq 0$.

The following examples show how these rules are used with rational expressions.

Study Tip

Alternative Method

When multiplying rational expressions, you can multiply first and then divide by the common factors. For instance, in Example 4, $\frac{4a}{5b} \cdot \frac{15b^2}{16a^3} = \frac{60ab^2}{80a^3b}$. Now divide the numerator and denominator by the common factors.

$$\frac{\overset{3}{\cancel{60}}\,\overset{1}{\cancel{a}}\,\overset{b}{\cancel{b^2}}}{\underset{4}{\cancel{80}}\,\underset{a^2}{\cancel{a^3}}\,\underset{1}{\cancel{b}}} = \frac{3b}{4a^2}$$

EXAMPLE — Multiply and Divide Rational Expressions

4 **Simplify each expression.**

a. $\frac{4a}{5b} \cdot \frac{15b^2}{16a^3}$

$$\frac{4a}{5b} \cdot \frac{15b^2}{16a^3} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{a}} \cdot 3 \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{b}} \cdot b}{\underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot \underset{1}{\cancel{a}} \cdot a \cdot a} \quad \text{Factor.}$$

$$= \frac{3 \cdot b}{2 \cdot 2 \cdot a \cdot a} \quad \text{Simplify.}$$

$$= \frac{3b}{4a^2} \quad \text{Simplify.}$$

b. $\frac{4x^2y}{15a^3b^3} \div \frac{2xy^2}{5ab^3}$

$$\frac{4x^2y}{15a^3b^3} \div \frac{2xy^2}{5ab^3} = \frac{4x^2y}{15a^3b^3} \cdot \frac{5ab^3}{2xy^2} \quad \text{Multiply by the reciprocal of the divisor.}$$

$$= \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{x}} \cdot x \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}}}{3 \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{a}} \cdot a \cdot a \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{y}} \cdot y} \quad \text{Factor.}$$

$$= \frac{2 \cdot x}{3 \cdot a \cdot a \cdot y} \quad \text{Simplify.}$$

$$= \frac{2x}{3a^2y} \quad \text{Simplify.}$$

CHECK Your Progress

4A. $\frac{8t^2s}{5r^2} \cdot \frac{15sr}{12t^3s^2}$

4B. $\frac{9m^2n^3}{16ab^4} \cdot \frac{8a^2b}{27m^5n}$

4C. $\frac{18ab^2}{25x^2y^3} \div \frac{9b}{10xy}$

4D. $\frac{14pq^2}{15w^7z^3} \div \frac{21p^3q}{35w^3z^8}$

Sometimes you must factor the numerator and/or the denominator first before you can simplify a product or a quotient of rational expressions.

EXAMPLE 5 **Polynomials in the Numerator and Denominator**

Simplify each expression.

a. $\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2}$

$$\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2} = \frac{(x + 4)\overset{1}{\cancel{(x - 2)}}}{(x + 3)\underset{1}{\cancel{(x + 1)}}} \cdot \frac{3\overset{1}{\cancel{(x + 1)}}}{\underset{1}{\cancel{(x - 2)}}}$$ Factor.

$$= \frac{3(x + 4)}{(x + 3)}$$ Simplify.

$$= \frac{3x + 12}{x + 3}$$ Distributive Property

b. $\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9}$

$$\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9} = \frac{a + 2}{a + 3} \cdot \frac{a^2 - 9}{a^2 + a - 12}$$ Multiply by the reciprocal of the divisor.

$$= \frac{(a + 2)\overset{1}{\cancel{(a + 3)}}\overset{1}{\cancel{(a - 3)}}}{\underset{1}{\cancel{(a + 3)}}(a + 4)\underset{1}{\cancel{(a - 3)}}}$$ Factor.

$$= \frac{a + 2}{a + 4}$$ Simplify.

CHECK Your Progress

5A. $\frac{y - 1}{5y + 15} \cdot \frac{y^2 + 5y + 6}{y^2 + 4y - 5}$

5B. $\frac{b^2 + 2b - 35}{b^2 - 4} \div \frac{b - 5}{b + 2}$

Simplify Complex Fractions

A **complex fraction** is a rational expression whose numerator and/or denominator contains a rational expression. The expressions below are complex fractions.

$$\frac{\frac{a}{5}}{3b} \qquad \frac{\frac{3}{t}}{t + 5} \qquad \frac{\frac{m^2 - 9}{8}}{\frac{3 - m}{12}} \qquad \frac{\frac{1}{p} + 2}{\frac{3}{p} - 4}$$

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To simplify a complex fraction, rewrite it as a division expression, and use the rules for division.

EXAMPLE 6 Simplify a Complex Fraction

6 Simplify $\dfrac{\frac{r^2}{r^2 - 25s^2}}{\frac{r}{5s - r}}$.

$$\frac{\frac{r^2}{r^2 - 25s^2}}{\frac{r}{5s - r}} = \frac{r^2}{r^2 - 25s^2} \div \frac{r}{5s - r}$$ Express as a division expression.

$$= \frac{r^2}{r^2 - 25s^2} \cdot \frac{5s - r}{r}$$ Multiply by the reciprocal of the divisor.

$$= \frac{\overset{1}{\cancel{r}} \cdot r(-1)\overset{1}{\cancel{(r - 5s)}}}{(r + 5s)\underset{1}{\cancel{(r - 5s)}}\underset{1}{\cancel{r}}}$$ Factor.

$$= \frac{-r}{r + 5s} \text{ or } -\frac{r}{r + 5s}$$ Simplify.

CHECK Your Progress

Simplify each expression.

6A. $\dfrac{\frac{(x + 3)^2}{x^2 - 16}}{\frac{x + 3}{x + 4}}$

6B. $\dfrac{\frac{y - 7}{y - 3}}{\frac{y^2 - 49}{y^2 + 4y - 21}}$

CHECK Your Understanding

Example 1 (pp. 442–443)

Simplify each expression.

1. $\dfrac{45mn^3}{20n^7}$

2. $\dfrac{a + b}{a^2 - b^2}$

3. $\dfrac{x^2 + 6x + 9}{x + 3}$

4. $\dfrac{36c^3d^2}{54cd^5}$

Example 2 (p. 443)

5. STANDARDIZED TEST PRACTICE Identify all values of y for which $\dfrac{y - 4}{y^2 - 4y - 12}$ is undefined.

A $-2, 4, 6$ B $-6, -4, 2$ C $-2, 0, 6$ D $-2, 6$

Simplify each expression.

Example 3 (p. 443)

6. $\dfrac{9y^2 - 6y^3}{2y^2 + 5y - 12}$

7. $\dfrac{b^3 - a^3}{a^2 - b^2}$

Example 4 (pp. 444–445)

8. $\dfrac{2a^2}{5b^2c} \cdot \dfrac{3bc^3}{8a^2}$

9. $\dfrac{3t + 6}{7t - 7} \cdot \dfrac{14t - 14}{5t + 10}$

10. $\dfrac{35}{16x^2} \div \dfrac{21}{4x}$

11. $\dfrac{20xy^3}{21} \div \dfrac{15x^3y^2}{14}$

Example 5 (p. 445)

12. $\dfrac{12p^2 + 6p - 6}{4(p + 1)^2} \div \dfrac{6p - 3}{2p + 10}$

13. $\dfrac{x^2 + 6x + 9}{x^2 + 7x + 6} \div \dfrac{4x + 12}{3x + 3}$

Example 6 (p. 446)

14. $\dfrac{\frac{c^3d^3}{a}}{\frac{xc^2d}{ax^2}}$

15. $\dfrac{\frac{2y}{y^2 - 4}}{\frac{3}{y^2 - 4y + 4}}$

Exercises

HOMEWORK HELP	
For Exercises	**See Examples**
16–19	1
20, 21	3
22–25	4
26–29	5
30–33	6
34, 35	2

Simplify each expression.

16. $\dfrac{30bc}{12b^3}$

17. $\dfrac{-3mn^3}{21m^2n^2}$

18. $\dfrac{5t-5}{t^2-1}$

19. $\dfrac{c+5}{2c+10}$

20. $\dfrac{3t-6}{2-t}$

21. $\dfrac{9-t^2}{t^2+t-12}$

22. $\dfrac{3xyz}{4xz} \cdot \dfrac{6x^2}{3y^2}$

23. $\dfrac{-4ab}{21c} \cdot \dfrac{14c^2}{18a^2}$

24. $\dfrac{3}{5d} \div \left(-\dfrac{9}{15df}\right)$

25. $\dfrac{p^3}{2q} \div \dfrac{-p}{4q}$

26. $\dfrac{3t^2}{t+2} \cdot \dfrac{t+2}{t^2}$

27. $\dfrac{4w+4}{3} \cdot \dfrac{1}{w+1}$

28. $\dfrac{4t^2-4}{9(t+1)^2} \cdot \dfrac{3t+3}{2t-2}$

29. $\dfrac{3p-21}{p^2-49} \cdot \dfrac{p^2-7p}{3p}$

30. $\dfrac{\frac{m^3}{3n}}{-\frac{m^4}{9n^2}}$

31. $\dfrac{\frac{p^3}{2q}}{-\frac{p^2}{4q}}$

32. $\dfrac{\frac{m+n}{5}}{\frac{m^2+n^2}{5}}$

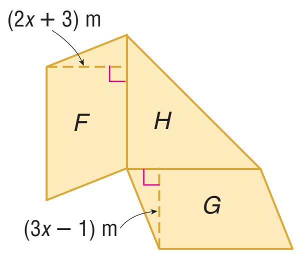
33. $\dfrac{\frac{x+y}{2x-y}}{\frac{x+y}{2x+y}}$

34. Under what conditions is $\dfrac{x-4}{(x+5)(x-1)}$ undefined?

35. For what values is $\dfrac{2d(d+1)}{(d+1)(d^2-4)}$ undefined?

36. **GEOMETRY** A parallelogram with an area of $6x^2 - 7x - 5$ square units has a base of $3x - 5$ units. Determine the height of the parallelogram.

37. **GEOMETRY** Parallelogram F has an area of $8x^2 + 10x - 3$ square meters and a height of $2x + 3$ meters. Parallelogram G has an area of $6x^2 + 13x - 5$ square meters and a height of $3x - 1$ meters. Find the area of right triangle H.



Simplify each expression.

38. $\dfrac{(-3x^2y)^3}{9x^2y^2}$

39. $\dfrac{(-2rs^2)^2}{12r^2s^3}$

40. $\dfrac{(-5mn^2)^3}{5m^2n^4}$

41. $\dfrac{y^2+4y+4}{3y^2+5y-2}$

42. $\dfrac{a^2+2a+1}{2a^2+3a+1}$

43. $\dfrac{3x^2-2x-8}{3x^2-12}$

44. $\dfrac{a^2-4}{6-3a}$

45. $\dfrac{b^2-4b+3}{3-2b-b^2}$

46. $\dfrac{6x^2-6}{14x^2-28x+14}$

47. $\dfrac{25a^2b^3}{6x^2y} \cdot \dfrac{8xy^2}{20a^3b^2}$

48. $\dfrac{-9cd}{8xw} \cdot \dfrac{(-4w)^2}{15c}$

49. $\dfrac{2x^3y}{z^5} \div \left(\dfrac{4xy}{z^3}\right)^2$

50. $\dfrac{w^2-11w+24}{w^2-18w+80} \cdot \dfrac{w^2-15w+50}{w^2-9w+20}$

51. $\dfrac{r^2+2r-8}{r^2+4r+3} \div \dfrac{r-2}{3r+3}$

52. $\dfrac{\frac{5x^2-5x-30}{45-15x}}{\frac{6+x-x^2}{4x-12}}$

EXTRA PRACTICE
See pages 907, 933.
Math Online
Self-Check Quiz at algebra2.com

Real-World Link

Ray Allen is a five-time All Star and member of team USA for the 2000 Olympics.

Source: NBA

53. Under what conditions is $\frac{a^2 + ab + b^2}{a^2 - b^2}$ undefined?

BASKETBALL For Exercises 54 and 55, use the following information.
At the end of the 2005-2006 season, the Seattle Sonics' Ray Allen had made 5422 field goals out of 12,138 attempts during his NBA career.

54. Write a ratio to represent the ratio of the number of career field goals made to career field goals attempted by Ray Allen at the end of the 2005-2006 season.

55. Suppose Ray Allen attempted a field goals and made m field goals during the 2006-2007 season. Write a rational expression to represent the ratio of the number of career field goals made to the number of career field goals attempted at the end of the 2006-2007 season.

AIRPLANES For Exercises 56–58, use the formula $d = rt$ and the following information.
An airplane is traveling at the rate r of 500 miles per hour for a time t of $(6 + x)$ hours. A second airplane travels at the rate r of $(540 + 90x)$ miles per hour for a time t of 6 hours.

56. Write a rational expression to represent the ratio of the distance d traveled by the first airplane to the distance d traveled by the second airplane.

57. Simplify the rational expression. What does this expression tell you about the distances traveled of the two airplanes?

58. Under what condition is the rational expression undefined? Describe what this condition would tell you about the two airplanes.

Graphing Calculator

For Exercises 59–62, consider $f(x) = \frac{-15x^2 + 10x}{5x}$ and $g(x) = -3x + 2$.

59. Simplify $\frac{-15x^2 + 10x}{5x}$. What do you observe about the expression?

60. Graph $f(x)$ and $g(x)$ on a graphing calculator. How do the graphs appear?

61. Use the table feature to examine the function values for $f(x)$ and $g(x)$. How do the tables compare?

62. How can you use what you have observed with $f(x)$ and $g(x)$ to verify that expressions are equivalent or to identify excluded values?

H.O.T. Problems

63. OPEN ENDED Write two rational expressions that are equivalent.

64. CHALLENGE Rewrite $\frac{a + \sqrt{b}}{-a^2 + b}$ so it has a numerator of 1.

65. Which One Doesn't Belong? Identify the expression that does not belong with the other three. Explain your reasoning.

$\frac{1}{x - 1}$	$\frac{x^2 + 3x + 2}{x - 5}$	$\frac{x + 1}{\sqrt{x + 3}}$	$\frac{x^2 + 1}{3}$

66. REASONING Determine whether $\frac{2d + 5}{3d + 5} = \frac{2}{3}$ is *sometimes*, *always*, or *never* true. Explain.

67. *Writing in Math* Use the information about rational expressions on page 442 to explain how rational expressions are used in mixtures. Include an example of a mixture problem that could be represented by $\frac{8 + x}{13 + x + y}$.

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STANDARDIZED TEST PRACTICE

68. ACT/SAT For what value(s) of x is $\frac{4x}{x^2 - x}$ undefined?

A $-1, 1$

B $-1, 0, 1$

C $0, 1$

D 0

69. REVIEW Which is the simplified form of $\frac{4x^3y^2z^{-1}}{(x^{-2}y^3z^2)^2}$?

F $\frac{4x^7}{y^4z^5}$

H $\frac{4}{y^3z^5}$

G $\frac{4xy}{z^5}$

J $\frac{4}{xy^4z^5}$

Spiral Review

Graph each function. State the domain and range. (Lesson 7-3)

70. $y = \sqrt{x - 2}$ **71.** $y = \sqrt{x} - 1$ **72.** $y = 2\sqrt{x} + 1$

73. Determine whether $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. (Lesson 7-2)

Determine whether each graph represents an odd-degree or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-3)

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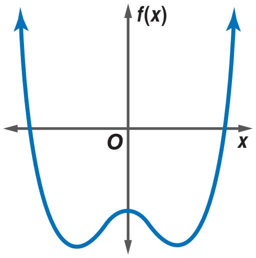

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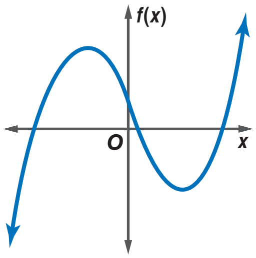

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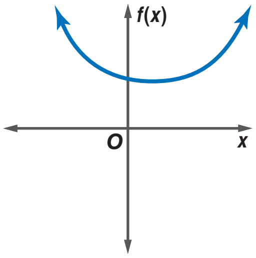

77. ASTRONOMY Earth is an average of 1.496×10^8 kilometers from the Sun. If light travels 3×10^5 kilometers per second, how long does it take sunlight to reach Earth? (Lesson 6-1)

Solve each equation by factoring. (Lesson 5-3)

78. $r^2 - 3r = 4$ **79.** $18u^2 - 3u = 1$ **80.** $d^2 - 5d = 0$

Solve each equation. (Lesson 1-4)

81. $|2x + 7| + 5 = 0$ **82.** $5|3x - 4| = x + 1$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 1-3)

83. $\frac{2}{3} + x = -\frac{4}{9}$ **84.** $x + \frac{5}{8} = -\frac{5}{6}$ **85.** $x - \frac{3}{5} = \frac{2}{3}$

86. $x + \frac{3}{16} = -\frac{1}{2}$ **87.** $x - \frac{1}{6} = -\frac{7}{9}$ **88.** $x - \frac{3}{8} = -\frac{5}{24}$